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Little string theory from double-scaling limits of field theories

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Abstract: We show that little string theory on S^5 can be obtained as double-scaling limits of the maximally supersymmetric Yang-Mills theories on $R \times S^2$ and $R \times S^3/Z_k$. By matching the gauge theory parameters with those in the dual supergravity solutions found by Lin and Maldacena, we determine the limits in the gauge theories that correspond to decoupling of NS5-brane degrees of freedom. We find that for the theory on $R \times S^2$, the 't Hooft coupling must be scaled like $\ln^3 N$, and on $R \times S^3/Z_k$, like $\ln^2 N$. Accordingly, taking these limits in these field theories gives Lagrangian definitions of little string theory on S^5.

Keywords: Gauge-gravity correspondence, Supersymmetric gauge theory.

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1. Introduction

Type IIA little string theory [1] describes the decoupling limit of NS5-branes in type IIA string theory in the limit where g_s is taken to zero at fixed α'. The remaining degrees of freedom are believed to be described by a non-gravitational six-dimensional theory. The infrared limit of this theory is known to be the $(0, 2)$ conformal field theory, but in general the theory is non-local (see [2] for a review).

Little string theory has a DLCQ formulation [3] as well as a deconstruction description [4], however it has mainly been analyzed through its gravity dual. This gravity dual is the near-horizon limit of the NS5-brane solution of type IIA string theory. For large r, where the IIA picture is valid, this is given by

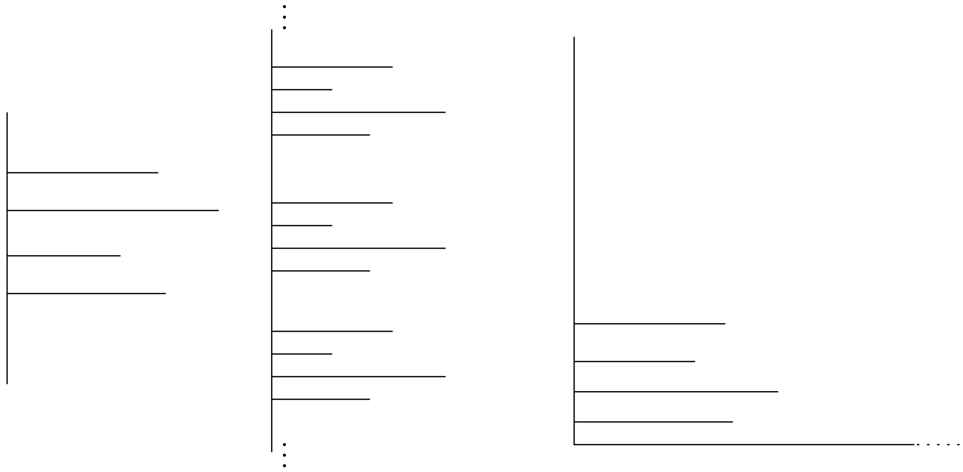
$$\begin{aligned} ds^2 &= N_5\alpha'(-dt^2 + d\vec{x}_5^2 + dr^2 + d\Omega_3^2) \\ e^\phi &= g_s e^{-r}, \end{aligned} \tag{1.1}$$

with N_5 units of H flux through the S^3. This description is also difficult to work with, however, since the linear dilaton sends the theory to strong coupling in the infrared region of the geometry.

Recently, Lin and Maldacena [5] found a supergravity solution in which the flat five-dimensional part of the geometry along the worldvolume of the NS5-branes is replaced with an S^5. For large radius this takes the form

$$\begin{aligned} ds^2 &= N_5\alpha'[2r(-dt^2 + d\Omega_5^2) + dr^2 + d\Omega_3^2] \\ e^\Phi &= g_s e^{-r}. \end{aligned} \tag{1.2}$$

This solution contains a linear dilaton and an S^3 with N_5 units of H-flux, implying we can think of this as describing NS5-branes on S^5. Interestingly, this supergravity solution has some features that make it more tractable than the solution corresponding to flat NS5-



(a) (b) (c)

Figure 1: The three generic types of electrostatics configurations. The isolated set of disks in (a) is a configuration dual to a vacuum of SYM theory on $R \times S^2$ with sixteen supercharges. The periodic configuration in (b) is dual to a vacuum of $\mathcal{N} = 4$ SYM theory on $R \times S^3/Z_k$. The set of disks above an infinite conducting plane in (c) is dual to a vacuum of the PWMM.

branes. The maximum values of the dilaton and the curvature are both tunable so that the supergravity description is valid everywhere.

This solution is one of a family of solutions of type IIA supergravity preserving SU(2|4) symmetry constructed by Lin and Maldacena [5]. Each supergravity solution is constructed from the electrostatic potential of an axisymmetric arrangement of charged conducting disks in three spatial dimensions. The above NS5-brane solution is obtained from the electrostatic potential between two infinitely large disks.

According to the proposal in [5], the supergravity solutions arising from configurations with a finite number of disks correspond to the (classically degenerate) vacua of super-Yang-Mills theory on $R \times S^2$ with sixteen supercharges. Configurations with an infinite number of disks, arranged in a periodic fashion, correspond to the vacua of $\mathcal{N} = 4$ super-Yang-Mills on $R \times S^3/Z_k$. Finally, configurations with one infinitely large disk, and a finite number of disks above it, correspond to the vacua of the plane wave matrix model (see figure 1). The relations among these field theories have been discussed in [6−8].

The supergravity picture suggests an interesting connection between these three gauge theories with SU(2|4) symmetry and little string theory. Lin and Maldacena showed that the supergravity solutions dual to the various vacua of these field theories generally contain throats with non-contractible S^3s permeated by H-flux, which can be associated with NS5-brane degrees of freedom. In the limit that the throats containing the non-contractible S^3s with H-flux become infinitely large, the NS5-brane degrees of freedom will decouple. The remaining geometry should be the above Lin-Maldacena NS5-brane solution dual to little string theory on S^5 [5]. In the language of the dual theories, this suggests that little string theory may be obtained from suitable limits of the three gauge theories.

In [9] the supergravity dual of a simple vacuum of the plane-wave model was considered, and the required limit that gives the Lin-Maldacena solution was explicitly determined. By matching the parameters of the plane wave matrix model with those of the electrostatics configuration, it was proposed that little string theory on S^5 may be obtained from a double-scaling limit of the plane wave matrix model.

In this paper, we extend the work of [9] and perform a similar analysis for the SYM theory on $R\times S^2$ and $\mathcal{N}=4$ SYM theory on $R\times S^3/Z_k$. We solve the electrostatics problems corresponding to specific simple vacua of these field theories and determine the scaling of parameters in the supergravity solutions that is required to obtain the Lin-Maldacena solution for NS5-branes on S^5. By considering the matching between the parameters in the field theories and those in the corresponding electrostatics problems, we thereby determine the precise scaling of the gauge theory parameters that is required to obtain little string theory on S^5. The proposed prescriptions are found to be double-scaling limits, similar to the one found in the case of the plane wave matrix model [9]. Whereas in the plane wave matrix model case it was found that the 't Hooft coupling must be scaled like $\ln^4 N$ [9], we will show below that for the SYM theories on $R\times S^2$ and $R\times S^3/Z_k$ the 't Hooft coupling must be scaled like $\ln^3 N$ and $\ln^2 N$ respectively.

2. The gauge theories and their dual supergravity solutions

In [5], Lin and Maldacena found a class of solutions of type IIA supergravity with SU(2|4) symmetry depending on one single function V. This function V solves the three dimensional Laplace equation and satisfies the same boundary conditions as the electrostatic potential of an axisymmetric arrangement of charged conducting disks in a background electric field. By specifying the positions and sizes of the conducting disks, the charges on the disks, and the asymptotic form of V at infinity, V is determined uniquely. Each different specification of these parameters leads to a different V, however not all such choices give rise to physically acceptable supergravity solutions. Flux quantization in the supergravity solution tells us that the charges on the disks and the spacing between disks are quantized. Positive-definiteness of various metric components in the supergravity solutions imposes constraints on the form of the asymptotic potential. Finally, the regularity of the supergravity solutions tells us that the surface charge density on the disks must vanish at the edge of the disks. This final condition suggests that for a fixed asymptotic potential, the positions, charges, and sizes of the disks cannot be independently specified. For example, the sizes of the disks may be fixed once the other parameters are freely specified. For an extensive discussion of the general properties of these supergravity solutions see [5].

Here we are interested in the supergravity solutions dual to the vacua of the SYM theory on $R\times S^2$ and $\mathcal{N}=4$ SYM theory on $R\times S^3/Z_k$. For all vacua of these two field theories, Lin and Maldacena determined the asymptotic form of V to be $W_0(r^2-2z^2)$, where $W_0>0$. The choice of vacuum is then given by specifying the charges, positions, and sizes of the disks. Let us review in some detail the connection between these parameters for the supergravity solutions and the parameters defining the field theory vacua.

First consider $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$. The space S^3/Z_k can be described most directly by choosing coordinates on the unit S^3 such that the metric takes the form

$$d\Omega_3^2 = \frac{1}{4}\left[(2d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right], \tag{2.1}$$

where the ψ coordinate is 2π periodic, and θ, ϕ are the usual coordinates for S^2. Then the orbifold S^3/Z_k is obtained by identifying $\psi \sim \psi + 2\pi/k$. The vacua of this field theory are given by the space of flat connections on S^3/Z_k. Up to gauge transformations, these are of the form $A = -\text{diag}(n_1, n_2, \ldots, n_N)\ d\psi$, where $e^{2\pi n_i/k}$ are k-th roots of unity (clearly, to label the vacua uniquely, we should restrict the values of the integers n_i to be in some fixed interval of length k). To understand intuitively how these vacua map to configurations of disks in the electrostatics problem, consider the field theory as a theory of D3-branes wrapped on an S^3/Z_k. Now apply a T-duality transformation in the isometry direction ψ. The T-dual coordinate $\tilde{\psi}$ is periodic $\tilde{\psi} \sim \tilde{\psi} + 2\pi k$, and the background gauge field is mapped to an arrangement of D2-branes located at the positions $\tilde{\psi} = 2\pi n_1, 2\pi n_2, \ldots, 2\pi n_N$ (along with their images under translations by integer multiples of $2\pi k$). Naturally, this suggests that the dual supergravity solution is obtained by considering a periodic configuration of disks with period proportional to k. The integers n_i that specify the gauge theory vacuum now determine the positions and charges of the disks within one period in the obvious manner. Presumably, the sizes of the disks are then fixed by demanding regularity of the supergravity solution. In rest of this paper, we will be interested in the simplest vacuum state of the theory, given by the trivial gauge field $n_i = 0$ for all i. In the normalization conventions of Lin and Maldacena [5], the dual supergravity solution is generated by the axisymmetric electrostatic potential $V(r, z)$ for an arrangement of equal-sized disks at $z = (\pi/2)km$ for all integers m, where the charge on each disk is $Q = (\pi^2/8)N$.

Now we consider the case of the SYM theory on $R \times S^2$. As discussed in [5], we can think of this theory as $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, in the limit where $k \to \infty$ and $g_{YM3}^2 \to 0$ while keeping $g_{YM3}^2 k$ fixed. Up to a numerical constant, the limiting value of $g_{YM3}^2 k$ is the coupling g_{YM2}^2. If we start with a vacuum in the S^3/Z_k theory with background gauge field $A = -\text{diag}(n_1, n_2, \ldots, n_N)d\psi$ and take $k \to \infty$ with the integers n_i fixed, then we obtain a vacuum of the S^2 theory with a vacuum expectation value for one of the adjoint scalars $\Phi = -\text{diag}(n_1, n_2, \ldots, n_N)$ and a background gauge field with associated flux $F = dA = \Phi \sin\theta d\theta d\phi$. All of the vacua of $\mathcal{N} = 4$ SYM on $R \times S^2$ discussed in [5] can be obtained in this way. This limit has a clear interpretation in the T-dual picture. We start with a configuration of a finite number of D2-branes, repeated periodically by translating the whole arrangement by integer multiples of $2\pi k$. In the limit $k \to \infty$, we are left with only one copy of the configuration of D2-branes, the images being pushed off to infinity. This naturally suggests that the dual supergravity solution is obtained by considering a configuration of a finite number of disks. It is clear that the integers n_i determine the positions and charges of the disks in a manner analogous to the situation in the $R \times S^3/Z_k$ theory. Again, the sizes of the disks are presumably fixed by demanding regularity of the supergravity solutions. Note that the total sum of the charge on the disks must equal the rank of the gauge group N. In the rest of this paper, we consider non-trivial vacua of the form $\Phi = (n, \ldots, n, -n, \cdots - n)$, where the integers n and $-n$ each appear $N/2$ times. In this case the dual supergravity solution is generated by the potential $V(r, z)$ corresponding to two equal-sized disks at $z = \pm(\pi/2)n$ with charge $(\pi^2/8)(N/2)$ on each disk.

The final issue we need to discuss in this section is the normalization of the asymptotic potential at infinity. For the SYM theory on $R \times S^2$, we can relate W_0 to g^2_{YM2} by using the results in [9]. As discussed in [6, 9, 8] the SYM theory on $R \times S^2$ can be obtained as a limit of the plane wave matrix model. This statement, together with the matching of parameters in the plane wave matrix model discussed in [9], tell us that we must have

$$W_0 = \frac{h_2}{g^2_{YM2}}, \tag{2.2}$$

where the positive constant h_2 does not depend on the parameters N, g^2_{YM2}, which define the gauge theory, and the eigenvalues of Φ, which label its vacua. For SYM theory on $R \times S^3/Z_k$, the above mentioned relation between this theory and the theory on $R \times S^2$ suggests that we make the identification

$$W_0 = \frac{h_3}{g^2_{YM3} k}, \tag{2.3}$$

where h_3 is a positive constant that does not depend on N, k, g^2_{YM3} nor the integers that label the vacua of the gauge theory.

3. Little string theory from SYM on $R \times S^2$

In this section, we consider in detail the supergravity solution corresponding to the electrostatics problem for two identical disks of radius R located at $z = \pm d$ with charge Q on each disk and a background potential $W_0(r^2 - 2z^2)$. We wish to solve the electrostatics problem explicitly and determine the required scaling to obtain the Lin-Maldacena NS5-brane solution.

The electrostatics problem for the case of two identical disks

Following the approach of [9], we first solve the electrostatics problem for the specific case $W_0 = 1$, $R = 1$, $d = \kappa$ (the solution for the general case is then obtained by linear rescaling of the coordinates and an overall rescaling of the potential). In this case the solution must have the form

$$V(r, z) = (r^2 - 2z^2) + \phi_\kappa(r, z), \tag{3.1}$$

where ϕ_κ is an axisymmetric solution of the Laplace equation that vanishes at infinity. We can expand ϕ_κ in terms of Bessel functions, and in the region between $z = -d$ and $z = d$, this expansion takes the form

$$\phi_\kappa(r, z) = \int_0^\infty \frac{du}{u} e^{-u\kappa} A(u) \big(e^{-uz} + e^{uz}\big) J_0(ru). \tag{3.2}$$

The potential on the two conducting disks, Δ, must be constant, and the electric field must be continuous at all points not on the disks. Imposing these boundary conditions leads to the following dual integral equations

$$\begin{aligned} \int_0^\infty \frac{du}{u}(1+e^{-2\kappa u})J_0(ru)A(u) &= \Delta - r^2 \qquad 0<r<1 \\ \int_0^\infty du J_0(ru)A(u) &= 0 \qquad r>1. \end{aligned} \tag{3.3}$$

Following [10] we find that the solution of these integral equations can be given in terms of the solution to a Fredholm integral equation of the second kind. The problem in this case is very similar to the one considered in [9]. We have

$$A(u) = \frac{2u}{\pi}\int_0^1 dt\cos(ut)f(t), \tag{3.4}$$

where $f(t)$ satisfies the integral equation

$$f(t) + \int_{-1}^1 dx K(t,x)f(x) = \Delta - 2t^2, \tag{3.5}$$

and

$$K(t,x) = \frac{1}{\pi}\frac{2\kappa}{4\kappa^2+(t-x)^2}. \tag{3.6}$$

For each value of Δ, the integral equation for f can be solved numerically. From the resulting electrostatics potential, we can compute the surface charge density on the disks

$$\sigma(r) = \frac{1}{\pi^2}\left[\frac{f(1)}{\sqrt{1-r^2}} - \int_r^1 dt\frac{f'(t)}{\sqrt{t^2-r^2}}\right]. \tag{3.7}$$

We can adjust the constant Δ until we find the value Δ_κ for which the corresponding solution f_κ satisfies $f_\kappa(1)=0$. Then the surface charge distribution $\sigma_\kappa(r)$ for this solution vanishes at the edge of the disks. This final condition ensures the regularity of the corresponding supergravity solutions. The total charge on each disk is given by

$$q_\kappa = \frac{2}{\pi}\int_0^1 dt f_\kappa(t). \tag{3.8}$$

Figure 3 shows a plot of q_κ. For large κ the charge on each disk approaches $8/3\pi$, and for small κ the charge on each disk approaches $4/3\pi$.

Finally the solution for the general case is obtained by rescaling. The electrostatics potential is given by

$$V(r,z) = W_0(r^2-2z^2) + W_0R^2\phi_{d/R}(r/R, z/R), \tag{3.9}$$

and the total charge on each disk is given by

$$Q = W_0R^3q_{d/R}. \tag{3.10}$$

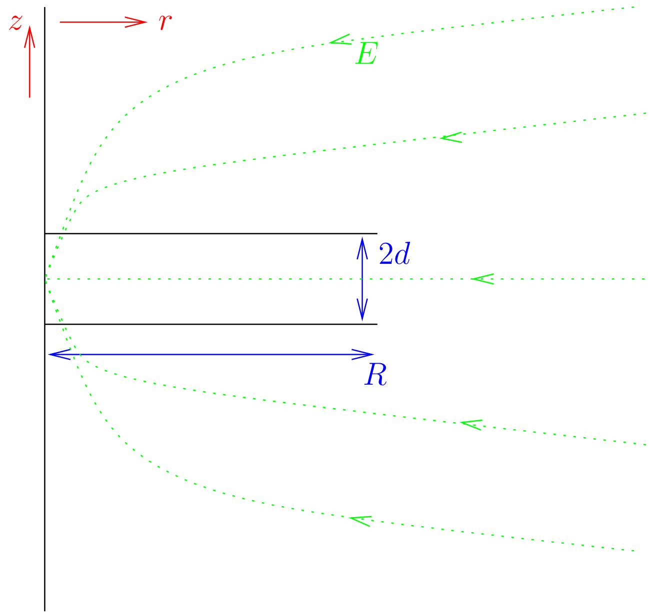


Figure 2: The electrostatics problem for two identical disks. The dotted lines show the background electric field configuration.

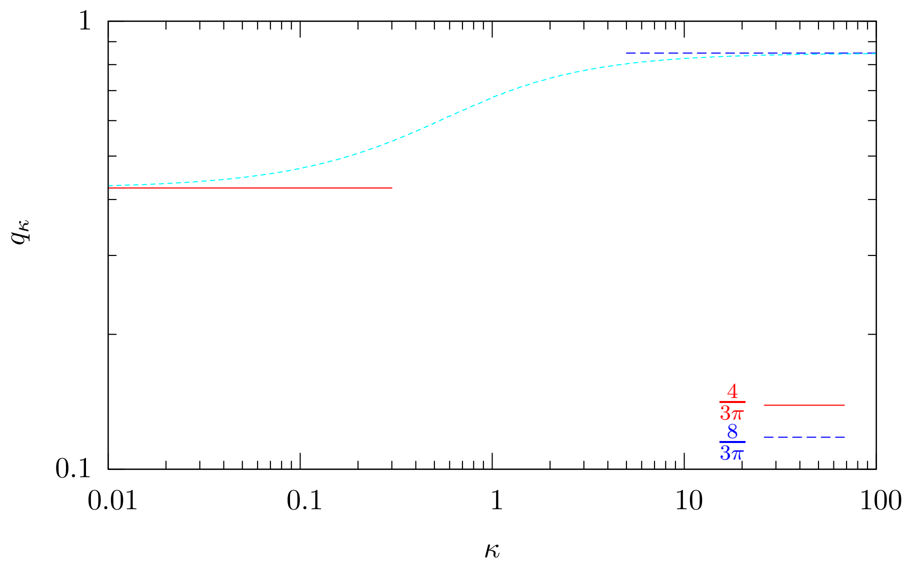


Figure 3: The charge on each disk in the two-disk case. The solid and dashed lines show the asymptotes for small and large κ respectively.

The limit of the Lin-Maldacena solution

Now we can determine the limit of this solution that gives the Lin-Maldacena solution for NS5-branes on S^5. In the region between the disks with $0 < r < R$, our solution is an axisymmetric solution of the Laplace equation that is regular at $r = 0$, so we can expand the solution in terms of modified Bessel functions

$$V(r,z) = V_{z=d} + \sum_{n=1}^{\infty} c_n \cos\left(\tfrac{(2n+1)\pi z}{2d}\right) I_0\left(\tfrac{(2n+1)\pi r}{2d}\right). \tag{3.11}$$

The coefficients c_n may be determined by using the potential at $r = R$. This gives

$$c_n = \left(I_0\left(\tfrac{(2n+1)\pi R}{2d}\right)\right)^{-1} 2W_0R^2 \int_0^1 dz \cos\left(\tfrac{(2n+1)\pi z}{2}\right)\left(1 - 2(\kappa z)^2 - \Delta_\kappa + \phi_\kappa(1,\kappa z)\right). \tag{3.12}$$

Using our numerical solution for ϕ_κ, the above integral can be performed numerically. In the limit $d \ll R$, this gives

$$c_1 \approx 1.56 W_0 R d \left(I_0\left(\tfrac{\pi R}{2d}\right)\right)^{-1}. \tag{3.13}$$

For large R/d this expression will be dominated by the Bessel function, which takes the asymptotic form

$$(I_0(z))^{-1} \sim \sqrt{2\pi z} e^{-z}$$

To preserve some non-trivial geometry, we must then scale W_0 exponentially. Doing so keeps c_1 finite in the limit, but sends all the other coefficients to zero so that we recover the Lin-Maldacena solution. More precisely, the Lin-Maldacena solution is obtained in the limit

$$R \to \infty \qquad d \text{ fixed} \qquad W_0 \sim R^{-1}(Rd)^{-1/2} e^{\frac{\pi R}{2d}}. \tag{3.14}$$

The gauge theory interpretation

Having understood the correct scaling on the gravity side, we can translate this into a condition on the gauge theory parameters. This amounts to

$$N \to \infty \qquad n \text{ fixed} \qquad \frac{1}{g_{YM2}^2}\lambda^{1/2} n^{1/2} e^{-b\lambda^{1/3}/n} \text{ fixed}, \tag{3.15}$$

where the 't Hooft coupling is $\lambda = g_{YM2}^2 N$ and b is a numerical coefficient related to the constant appearing in (2.2) by $b = (\pi/4)(3/h_2)^{1/2}$. We see that this is a large N limit, where the 't Hooft coupling is also scaled to infinity in a controlled way, and is very similar to the limit that was found in the case of the PWMM in [9]. Note that the number of NS5-branes is $N_5 = 2n$.

4. Little string theory from $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$

Now we wish to perform a similar detailed analysis for the supergravity solution corresponding to a periodic array of disks of radius R, where the disks are located at $z = (2m+1)d$ (m is any integer), the charge on each disk is Q, and the background electric field is given by the potential $W_0(r^2 - 2z^2)$. Again we first solve the electrostatics problem, then find the limit that recovers the NS5-brane solution of Lin and Maldacena.

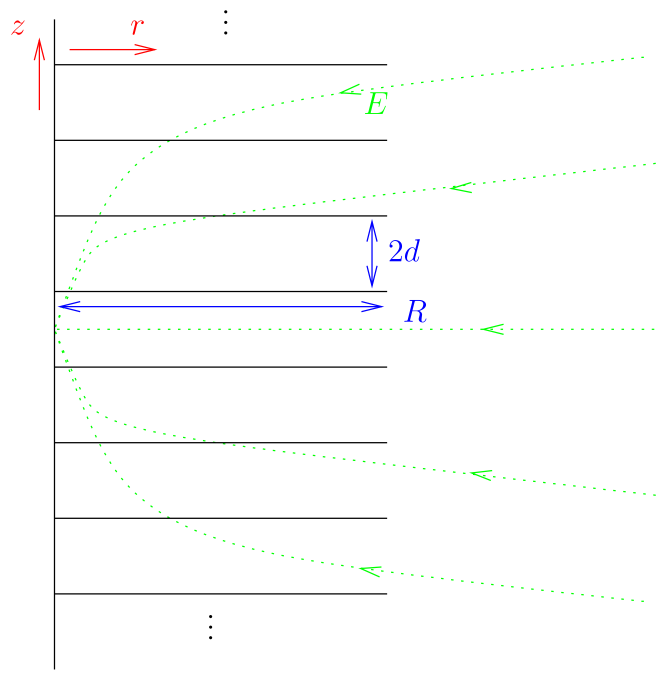


Figure 4: The electrostatics problem in the case a periodic array of disks. The dotted lines show the background electric field configuration.

The electrostatics problem for a periodic array of disks

As in the previous section, we solve the electrostatics problem for the special case $W_0 = 1$, $R = 1$, $d = \kappa$, and obtain the solution for the general case by rescaling. In the absence of the background potential the charge distribution on each disk will be the same. Adding the background field will affect the charge distribution on each disk, but since the radial part of the electric field it creates is identical on each disk, the charge distribution will remain the same on each disk (see figure 4).

We can separate the potential into the sum of the background field and the part due to the charge on the disks.

$$V = r^2 - 2z^2 + \phi_\kappa(r, z), \tag{4.1}$$

where ϕ_κ is periodic in z because the charge on each disk is identical. Formally, we can expand $\phi_\kappa(r, z)$ in terms of Bessel functions as

$$\phi_\kappa(r, z) = \int_0^\infty \frac{du}{u} J_0(ru)A(u) \sum_{n=-\infty}^{\infty} e^{-u|(2n+1)\kappa - z|}, \tag{4.2}$$

and then try to determine the function $A(u)$ by the imposing the boundary conditions. If we take the value of the potential V to be $\Delta - 2\kappa^2$ on the disk at $z = \kappa$, then by imposing the boundary conditions we obtain the following dual integral equations

$$\begin{aligned} \int_0^\infty \frac{du}{u} \left(1 + \frac{2e^{-2\kappa u}}{1 - e^{-2\kappa u}}\right) J_0(ru)A(u) &= \Delta - r^2 \qquad 0 < r < 1 \\ \int_0^\infty du J_0(ru)A(u) &= 0 \qquad r > 1. \end{aligned} \tag{4.3}$$

However, direct attempts to solve these equations are met with divergences and various difficulties. The reason is that these equations hold only formally, because the sum in the expression for the potential (4.2) actually diverges. Physically, there is no divergence because the electric field remains finite. This is the same type of situation encountered for an infinite number of equally spaced point charges (or an infinite line of charge) on the z-axis, which occurs simply because we try to express the potential as a sum of the Coulomb potential for each charge. If we consider the potential difference between any two points, there is no divergence, so we can regularize (4.2) by subtracting the potential at any fixed reference point.

In this case, it is more convenient to consider the first integral equation (4.3) as a condition on the electric field rather than the electric potential

$$\int_0^\infty du \left(1 + \frac{2e^{-2\kappa u}}{1 - e^{-2\kappa u}}\right) J_1(ru)A(u) = 2r \qquad 0 < r < 1. \tag{4.4}$$

The dual integral equations can then be solved by introducing a function satisfying a Fredholm integral equation of the section kind,

$$f_\kappa(x) + \int_0^1 duK(x,u)f_\kappa(u) = -\frac{8x}{\sqrt{\pi}}, \tag{4.5}$$

where

$$A(u) = -\frac{1}{\sqrt{\pi}} \int_0^1 d\xi \, \sin(u\xi) f_\kappa(\xi). \tag{4.6}$$

The kernel is given by

$$K(x,u) = \frac{1}{\pi} \int_0^\infty dtk(t)(-\cos(u+x)t + \cos|u-x|t), \tag{4.7}$$

where

$$k(u) = \frac{2e^{-2\kappa u}}{1 - e^{-2\kappa u}}. \tag{4.8}$$

These integrals can be evaluated and the result is

$$\begin{aligned} K(x,u) = \frac{1}{2\pi\kappa}\Big(\Psi\Big(1 + \frac{i(x+u)}{2\kappa}\Big) + \Psi\Big(1 - \frac{i(x+u)}{2\kappa}\Big) \\ - \Psi\Big(1 + \frac{i|x-u|}{2\kappa}\Big) - \Psi\Big(1 - \frac{i|x-u|}{2\kappa}\Big)\Big), \end{aligned} \tag{4.9}$$

where Ψ is the digamma function. We solved (4.5) numerically using the Nyström method (e.g. [11]). In contrast to the two disk case, since we considered the integral equation corresponding to a condition on the electric field, there is no Δ to adjust to ensure that the surface charge density at the edge of the disk vanishes. In fact, for the form of the solution given in (4.6), this condition is automatically satisfied as long as f_κ is bounded. In terms of f_κ, the charge on each disk is

$$q_\kappa = -\frac{1}{\sqrt{\kappa}} \int_0^1 dt \; t f_\kappa(t). \tag{4.10}$$

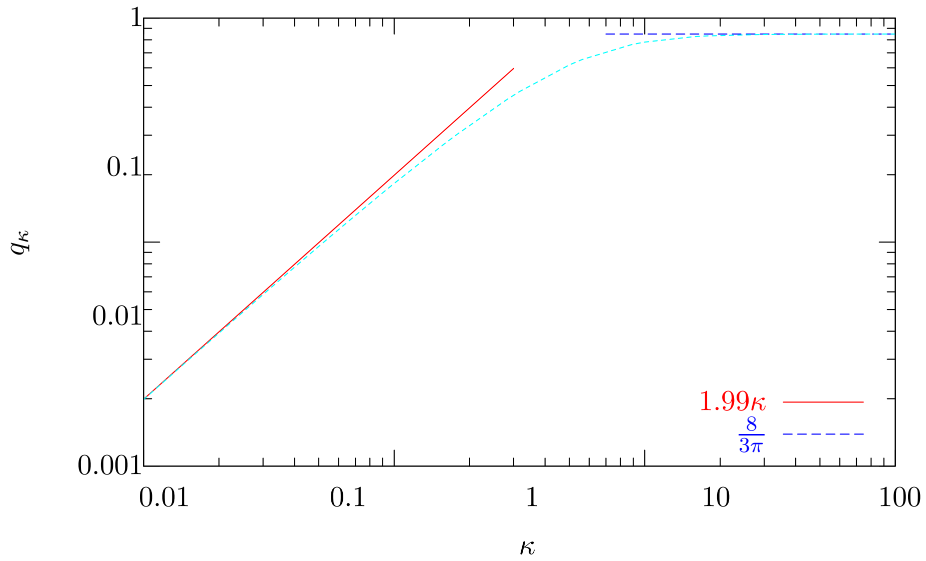


Figure 5: The charge on a disk as a function of the spacing between disks. The numerical result is given by the dashed line. The solid line is the asymptotic behaviour for small κ, $q \sim 1.99\kappa$. For large κ the charge approaches $\frac{8}{3\pi}$.

Using our numerical solution for f_κ we found that q_κ approaches $8/3\pi$ for large κ and approximately 1.99κ for small κ (see figure 4).

In principle, it is possible to determine the regularized potential completely from this solution for f_κ (however, the integrals involved are rather computationally expensive). Then the potential for the case of general W_0, R and d is obtained by a linear rescaling of coordinates and an overall rescaling of the potential. Specifically, we note that the charge on each disk in the general case is

$$Q = W_0 R^3 q_{d/R}. \tag{4.11}$$

The limit of the Lin-Maldacena solution

To determine how the Fourier coefficients of the potential scale with κ, we found it was most efficient to use the method of conformal mapping. Near the edge of the disks, when their radial size is much larger than their separation, the electrostatics problem becomes two-dimensional. By defining the complex coordinates $\zeta = (r - R) + iz$, and $w = 2\partial_\zeta V$, any holomorphic function $w(\zeta)$ will be a solution of the Laplace equation. As described in [5] the appropriate mapping in this case is

$$\partial_w \zeta = \alpha \tanh\left(\frac{\pi w}{\beta}\right) \tag{4.12}$$

and so

$$\zeta = \frac{\alpha\beta}{\pi} \log\cosh\left(\frac{\pi w}{\beta}\right), \tag{4.13}$$

where α, β are constants. Inverting this we find

$$w = \frac{\beta}{\pi} \cosh^{-1} \left(e^{\frac{\pi\zeta}{\alpha\beta}} \right). \tag{4.14}$$

If we fix the positions of the disks to be at $\zeta = i(2md)$, where m is an integer, we have $d = \alpha\beta/2$. The vertical electric field at any disk should be $-4W_0\Im(\zeta)$, so that $\beta = 8W_0 d$ and $\alpha = 1/4W_0$.

Expanding the potential in terms of modified Bessel functions, as in the two-disk case, we find that

$$c_1 \approx \frac{16W_0 d^2}{\pi}(I_0(\tfrac{\pi R}{2d}))^{-1}(0.659). \tag{4.15}$$

Again, therefore, to preserve non-trivial geometry we must scale W_0 exponentially. The precise scaling form to obtain the Lin-Maldacena solution is

$$R \to \infty \qquad\qquad d \text{ fixed} \qquad\qquad W_0 \sim R^{-1/2} d^{-3/2} e^{\frac{\pi R}{2d}}. \tag{4.16}$$

The gauge theory interpretation

In terms of the gauge theory parameters, we have

$$N \to \infty \qquad k \text{ fixed} \qquad \frac{1}{g^2_{YM3}} \lambda^{1/4} k^{1/2} e^{-c\lambda^{1/2}/k} \text{ fixed}, \tag{4.17}$$

where the 't Hooft coupling is $\lambda = g^2_{YM3}N$ and c is a numerical coefficient related to the constant appearing in (2.3) by $c = (2\pi/1.99h_3)^{1/2}$. This is again a double-scaling limit in which the 't Hooft coupling is scaled to infinity in a controlled way. Note that the number of NS5-branes in this case is $N_5 = k$.

5. Discussion

We have given an explicit prescription for taking double-scaling limits of SYM theory on $R \times S^2$ and $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$ to obtain little string theory on S^5. These limits were obtained by using the family of supergravity solutions found by Lin and Maldacena [5]. With the similar result in [9], we have demonstrated that it is possible to take such a limit in each of the three generic examples of this family of solutions, and in each of the three field theories to which they are dual.

In each case, the precise form of the double-scaling limit is similar. Whereas in the plane wave matrix model it was found the correct limit was [9]

$$N_2 \to \infty \qquad N_5 \text{ fixed} \qquad N_2 \sim \lambda^{5/8} e^{a\lambda^{1/4}/N_5}, \tag{5.1}$$

we found above that for the SYM theory on $R \times S^2$ we have

$$N \to \infty \qquad n \text{ fixed} \qquad N \sim \lambda^{1/2} n^{-1/2} e^{b\lambda^{1/3}/n}, \tag{5.2}$$

and for $\mathcal{N} = 4$ SYM theory on $R \times S^3/Z_k$ we have

$$N \to \infty \qquad k \text{ fixed} \qquad N \sim \lambda^{3/4} k^{-1/2} e^{c\lambda^{1/2}/k}. \tag{5.3}$$

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As noted in [9], it is sensible that the correct limit to obtain little string theory from these field theories is a double-scaling limit as opposed to a strict 't Hooft limit. If the correct limit was the 't Hooft limit, then it would seem strange that the field theory could produce string loop interactions. That the 't Hooft coupling should also be scaled to infinity in a controlled way allows the field theory to reproduce the string genus expansion.

Suppose we consider the genus expansion for some physical observable in one of these theories

$$F = \sum_g N^{2-2g} f_g(\lambda, \alpha), \tag{5.4}$$

where α represents the other parameters. The double-scaling limit should be such that all terms in this expansion contribute. For this to occur, the terms in the expansion would have to take a particular form when λ is large. In the case of the PWMM, this form was found to be [9]

$$f_g(\lambda) \to a_g \big(\lambda^{5/8} e^{a\lambda^{1/4}/N_5}\big)^{2g-2}, \tag{5.5}$$

where the bracketed expression divided by N_2 serves as the effective coupling constant. Here we find for the SYM theory on $R \times S^2$ we must have

$$f_g(\lambda) \to a_g \big(\lambda^{1/2} e^{b\lambda^{1/3}/n}\big)^{2g-2}, \tag{5.6}$$

and for $\mathcal{N} = 4$ SYM theory on $R \times S^3/Z_k$

$$f_g(\lambda) \to a_g \big(\lambda^{3/4} e^{c\lambda^{1/2}/k}\big)^{2g-2}. \tag{5.7}$$

Interestingly, although these field theories live in different numbers of dimensions, it is possible to recover little string theory from each of them by similar double-scaling limits.

Obvious difficulties arise in checking these predictions. One might hope that there are some BPS observables for which such a check might be feasible. In the case of the circular Wilson loop in $\mathcal{N} = 4$ SYM the full set of planar diagrams can be summed [12]. The result in that case took the form

$$\langle W \rangle_{N=\infty} = \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}}. \tag{5.8}$$

This result has been extended to all orders in [13], where it was shown that the asymptotic behaviour goes like $e^{\sqrt{\lambda}}$ at each order. That behaviour also arises from modified Bessel functions. It would be interesting to calculate the circular Wilson loop in $\mathcal{N} = 4$ SYM on $R \times S^3/Z_k$, and to compare it with our results here.

Other open questions remain. For example, as noted in [9], the solution for little string theory on S^5 given by Lin and Maldacena [5] is the simplest of an infinite family of solutions that have an infinite throat with H-flux. It would be interesting to understand if these solutions could arise from limits of more general disk configurations. It would also be interesting to understand more about the vacua of little string theory dual to these solutions.

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